

# Hyper-reguli in $\text{PG}(5, q)$

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## Abstract

A simple counting argument is used to show that for all  $q$ , an André hyper-regulus  $\mathbb{X}$  in  $\text{PG}(5, q)$  has exactly two switching sets. Moreover, there are exactly  $2(q^2 + q + 1)$  planes in  $\text{PG}(5, q)$  that meet every plane of  $\mathbb{X}$  in a point, namely the planes in the switching sets.

Bruck [1] investigated covers of the circle geometry  $CG(3, q)$  and showed a cover corresponds to a set  $\mathbb{X}$  in  $\text{PG}(5, q)$ . The set  $\mathbb{X}$  consists of  $q^2 + q + 1$  mutually disjoint planes, and has two *switching sets*  $\mathbb{Y}, \mathbb{Z}$ , with the property that two planes from different sets ( $\mathbb{X}$ ,  $\mathbb{Y}$  or  $\mathbb{Z}$ ) meet in a unique point, and two planes from the same set are disjoint. Ostrom [2] called these three sets *André hyper-reguli*, and posed the question of whether there are any other hyper-reguli that cover the same set of points as  $\mathbb{X}$ . Pomareda [3] used algebraic techniques to show that there are no other switching sets of  $\mathbb{X}$  when the largest power of 3 that divides  $q - 1$  is 1. In this article we prove that this result holds for all  $q$  using a simple counting argument. Moreover, we show that not only are there no more hyper-reguli for  $\mathbb{X}$ , but there are no further planes in  $\text{PG}(5, q)$  that meet each plane of  $\mathbb{X}$  in a point.

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**Lemma 1** *The number of covers of the circle geometry  $CG(3, q)$  is  $\frac{1}{2}q^3(q-1)(q^3+1)$ .*

**Proof** In [1], it is shown that every cover of  $CG(3, q)$  can be represented as either I:  $\{x \in \text{GF}(q^3) : N(x-a) = f\}$ , for some  $a \in \text{GF}(q^3)$ , and  $f \in \text{GF}(q) \setminus \{0\}$ ; or II:  $\{x \in \text{GF}(q^3) \cup \{\infty\} : N(\frac{x-a}{x-b}) = f\}$ , for some  $a \neq b \in \text{GF}(q^3)$ , and  $f \in \text{GF}(q) \setminus \{0\}$ , where  $N$  is the norm from  $\text{GF}(q^3)$  to  $\text{GF}(q)$ , that is  $N(x) = x^{q^2+q+1}$ . We count the number of covers, firstly, there are  $q^3(q-1)$  covers of type I. To count covers of type II, note that if  $N((x-a)/(x-b)) = f$ , then  $N((x-b)/(x-a)) = 1/f$ . So the number of covers of type II is  $\frac{1}{2}q^3(q^3-1)(q-1)$ .  $\square$

**Theorem 2** *Let  $\mathbb{X}$  be an André hyper-regulus in  $\text{PG}(5, q)$ . Then there are exactly  $2(q^2 + q + 1)$  planes that meet every plane of  $\mathbb{X}$ , namely the planes in the two switching sets of  $\mathbb{X}$ .*

**Proof** Let  $\mathcal{S}$  be a regular 2-spread containing  $\mathbb{X}$ . By [1], the planes of  $\mathcal{S}$  correspond to points of the circle geometry  $CG(3, q)$ , and the covers of  $CG(3, q)$  are equivalent to the André hyper-reguli contained in  $\mathcal{S}$ . Moreover by [1], given an André hyper-reguli  $\mathbb{X}$  contained in  $\mathcal{S}$ , there are at least  $2(q^2 + q + 1)$  planes that meet every plane of  $\mathbb{X}$  in a point (namely the planes in the two switching sets). The number of covers of  $CG(3, q)$  is counted in Lemma 1, hence the number  $x$  of planes of  $\text{PG}(5, q)$  that meet  $q^2 + q + 1$  planes of  $\mathcal{S}$  in a point is at least  $y = \frac{1}{2}q^3(q-1)(q^3+1) \times 2(q^2 + q + 1)$ .

We now count the number  $x$  exactly. As the 2-spread  $\mathcal{S}$  covers all the points of  $\text{PG}(5, q)$ , we can partition the planes of  $\text{PG}(5, q)$  into three types with respect to  $\mathcal{S}$ . Type A consists of planes of  $\mathcal{S}$ ; Type B consists of planes that meet  $q^2 + q + 1$  elements of  $\mathcal{S}$  in exactly one point; and Type C consists of planes that meet one element of  $\mathcal{S}$  in a line (and so meet  $q^2$  elements of  $\mathcal{S}$  in exactly one point). We note that planes of type B and C determine linear sets of  $\text{PG}(1, q^3)$  of rank 3. We count the number of planes of each type. There are  $q^3 + 1$  planes of type A. To count planes of type C, note that there are  $q^3 + 1$  choices for a plane of  $\mathcal{S}$ , each contains  $q^2 + q + 1$  lines, and each of these lies in  $q^3 + q^2 + q$  planes not in  $\mathcal{S}$ . Hence there are  $q(q^3 + 1)(q^2 + q + 1)^2$  planes of type C. The total number of planes in  $\text{PG}(5, q)$  is  $(q^3 + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)$ . Hence the remaining  $x = q^3(q^3 + 1)(q^3 - 1)$  planes are of type B.

Thus we have  $x = y$ , that is, each plane of  $\text{PG}(5, q)$  that meets  $q^2 + q + 1$  planes of  $\mathcal{S}$  lies in an André hyper-regulus of  $\mathcal{S}$ , or in one of the two known switching sets of an André hyper-regulus of  $\mathcal{S}$ . Thus there are no other planes of  $\text{PG}(5, q)$  that meet each plane of an André hyper-regulus.  $\square$

## References

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